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Anomalous hydrodynamics

George M. Newman*

Institute for Nuclear Theory, University of Washington
Seattle, WA 98195-1550, U.S.A.
E-mail: gmnewman@phys.washington

Abstract: Our goal is to examine the role of anomalies in the hydrodynamic regime of field theories. We employ methods based on gauge/gravity duality to examine R-charge anomalies in the hydrodynamic regime of strongly t'Hooft coupled, large-N, $\mathcal{N}=4$ Super Yang-Mills. We use a single particle spectrum treatment based on the familiar "level-crossing" picture of chiral anomalies to investigate thermalized massless QED. In each case we work in the presence of a homogeneous background magnetic field, and find the same result. Regardless of whether a particular current is anomalously non-conserved or not, as long as it participates in an anomalous 3-pt. correlator, its constitutive relation receives a new term: $\mathbf{j}^a \propto -d^{abc}\mathbf{B}^b\rho^c$. This agrees with results found by Alekseev et.al. for QED. We include a general, symmetry based argument for the presence of such terms, and use linear response theory to determine their coefficients in a model with anomalous global charges. This last method, we apply to briefly examine baryon transport in chiral QCD in a strong magnetic field.

Keywords: QCD, Thermal Field Theory, AdS-CFT Correspondence, Anomalies in Field and String Theories.

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Contents

1. Introduction

Hydrodynamics [1] provides a universal description of the long range, long time-scale behavior of a wide variety of thermal systems. The hydrodynamic quantities are those for which small perturbations from equilibrium have relaxation times which diverge as their wavelength diverges. As a result, hydrodynamics has been formulated as a classical effective field theory with various fields, typically the conserved quantities, playing the role of fundamental degrees of freedom. Hydrodynamic behavior has been seen to arise from finite temperature quantum field theory [2].

At the same time however, one feature that sets aside relativistic quantum field theories is the existence of quantum anomalies, which cause some classically conserved quantities to be non-conserved. An unsolved problem is how to incorporate the effects of quantum anomalies into the hydrodynamics description of a thermal field theory which contain anomalies. In a QCD plasma, the hydrodynamic theory should reproduce the three-point correlation functions including the anomalous part (an example of such correlation functions is that of the axial vector, vector and baryon currents). Another example is the massless QED plasma (or the QED plasma at such high temperatures so that the mass of the electron can be neglected). In this case, magnetohydrodynamics has to be enlarged to incorporate the axial current $j^{5\mu} = \bar{\psi}\gamma^\mu\gamma^5\psi$. Although this current is not conserved, $\partial_\mu j^{5\mu} \sim \mathbf{E}\cdot\mathbf{B}$, at large distances the total axial charge should change slowly because the conducting plasma cannot support a large long-distance electric field $\mathbf{E}$.

In this paper, we consider a simpler problem where anomalies enter the hydrodynamic equations already at the linearized level. Namely, we consider a theory with a set of global conserved charges j^a for which the triple correlators contain anomalous contributions. We then turn on an external magnetic field coupled to one of the global charges. An example of such a system is high-temperature QCD in a very large magnetic field. Now anomalies appear already at the level of two-point correlators, and hence should be manifested in the linearized hydrodynamic equations.

We shall argue that in the presence of a background magnetic field coupled to some conserved charge, the constitutive equation for the currents is modified in the presence of quantum anomalies. In an anomalous theory, any current j^a participating in an anomaly now receives a contribution

$$\mathbf{j}^a \propto d^{abc}\mathbf{B}^b\rho^c. \tag{1.1}$$

So, in addition to the diffusion and Ohmic current, there is a new, dissipationless contribution proportional to the magnetic field and a density of charges. It is important to note that this modification of the constitutive relation occurs regardless of whether the current itself is anomalously non-conserved. For example, in the massless QED mentioned above, both the axial vector current and the electromagnetic vector current receive this new term in their constitutive equations. This result has been obtained previously, by Alekseev, Cheianov, and Frohlich, for the cases of 2D field theory, and massless QED [3]. In this work, we will confirm their result, using different methods, as well as extending our analysis to a wider variety of systems.

We use two complimentary approaches to elucidate the impact of quantum anomalies on the hydrodynamic regime. The first approach uses the gauge/gravity duality, in which the hydrodynamic behavior of currents in 4D thermal gauge theories is obtained from the dynamics of the dual Yang-Mills fields on black-brane backgrounds in higher-dimensions. In this approach the 4D quantum anomaly has a very simple dual description in the higher-dimensional theory — it corresponds to the 5D Chern-Simons term in the gauge action. This approach applies only to strongly coupled gauge theories with gravity duals, in particular, to $\mathcal{N} = 4$ super-Yang-Mills theory in the large-N, large t'Hooft coupling regime. In the second approach, which is appropriate in the weak-coupling regime, we use a single particle spectrum treatment, similar to the "level-crossing" pictures usually used for explaining anomalies [4], to examine the behavior of currents in a weakly coupled abelian gauge theory with axial anomaly. In both cases we perform our analysis in the background of a constant, homogeneous magnetic field and obtain *the same* additional term in the constitutive relation for an anomalous current. We then argue that the form of the new term in the constitutive equation that we found is exact, i.e., independent of the strength of interactions.

The paper is organized as follows. In the first section, we use simple symmetry arguments to motivate the inclusion of terms of the form in [3] in the hydrodynamics of anomalous theories. In the second section, we explore the R-charge anomaly in the hydrodynamic regime of $\mathcal{N} = 4$ SYM using a "membrane paradigm" treatment of the dual 5D theory which is very similar to that used in [5] to examine non-anomalous hydrodynamics in theories with gravitational duals. In the third section we present a quasiparticle analysis of the axial anomaly in massless QED at finite T and finite (electron) chemical potential. Next we present our argument for the universality of this result. This argument is based on the equilibrium form of a charge distribution in a system with non- vanishing gauge field. In section four, we consider the application of our results to QCD at high temperature and strong magnetic field. Then, we conclude.

2. Symmetry considerations

The simplest model where the problem of anomalous hydrodynamics appears is QCD with two massless quark flavors. The conserved currents in the theory are the isospin current $j^{a\mu} = \bar{q}\gamma^\mu \frac{\tau^a}{2} q$, the axial isospin current $j^{a\mu} = \bar{q}\gamma^\mu\gamma^5 \frac{\tau^a}{2} q$, and the and the baryon current $j^\mu_B = \bar{q}\gamma^\mu q$. We turn on a background magnetic field $\mathbf{B}$ coupled to the baryon current and discuss the hydrodynamic behavior of the isospin and the axial isospin currents.

In the absence of the magnetic field $\mathbf{B}$ and at temperatures higher than the chiral phase transition, the hydrodynamic equations for the vector and axial charge densities ρ^a and ρ^{5a} are the diffusion equations which can be written as the conservation laws

$$\dot{\rho}^a + \boldsymbol{\nabla}\cdot\mathbf{j}^a = 0, \qquad \dot{\rho}^{5a} + \boldsymbol{\nabla}\cdot\mathbf{j}^{5a} = 0, \tag{2.1}$$

coupled with the constitutive relations

$$\mathbf{j}^a = -D\boldsymbol{\nabla}\rho^a, \qquad \mathbf{j}^{5a} = -D\boldsymbol{\nabla}\rho^{5a} \tag{2.2}$$

We recall that the form of the constitutive equations are dictated by the symmetries (rotational, C and P) and by the fact that we limit ourselves, in linearized hydrodynamics, to terms linear in fields with the lowest number of spatial derivatives.

However, in the presence of the external magnetic field $\mathbf{B}$, it is possible to write additional linear terms

$$\mathbf{j}^a = -D\boldsymbol{\nabla}\rho^a + c\mathbf{B}\rho^{5a}, \qquad \mathbf{j}^{5a} = -D\boldsymbol{\nabla}\rho^{5a} + c'\mathbf{B}\rho^a \tag{2.3}$$

The equations are obviously rotationally invariant. To see that they respect C and P invariance one recalls that under C $j^\mu \to -j^\mu$, $j^{5\mu} \to j^{5\mu}$, and $\mathbf{B} \to -\mathbf{B}$, and under P $\rho \to \rho$, $\mathbf{j} \to -\mathbf{j}$, $\rho^5 \to -\rho^5$, $\mathbf{j}^5 \to \mathbf{j}^5$, and $\mathbf{B} \to \mathbf{B}$.

The coefficients c and c' are not fixed by symmetries. In the next two sections we compute these coefficients in some simple theories.

3. An approach from gauge/gravity duality

3.1 Introduction

In this section we follow an analysis developed in ref. [5], which employed the gauge/gravity duality and the black-hole "membrane paradigm" to demonstrate hydrodynamic behavior in a variety of finite-temperature theories with holographic gravitational duals. We will work with the particular case of an $\mathcal{N} = 4$ super-Yang-Mills theory, believed dual to a stack of black D3 branes in $AdS_5 \times S^5$. (The hydrodynamic behavior of this theory has been found directly.) For the sake of completeness we will repeat many of the arguments used in ref. [5]. This repetition also serves to demonstrate explicitly where our assumptions do or do not differ from those of the previous work.

The AdS_5 metric for our black brane configuration is,

$$ds_5^2 = \frac{(\pi T R)^2}{u}\left[-(1-u^2)dt^2 + \sum_{i=1}^{3} dx_i^2\right] + \frac{R^2}{4u^2(1-u^2)}du^2. \tag{3.1}$$

Note the use of the dimensionless radial coordinate, u, which is related to the usual radial coordinate r and horizon location r_0 through $u = r_0^2/r^2$. The theory possesses SU(4) conserved R-charges, whose correlation functions are computable using AdS/CFT. The the 5D bulk action of the gauge field A_μ^a dual to the R-current, complete with Chern-Simons term is,

$$S = -\frac{1}{4g_{\rm GS}^2}\int d^5x\,\sqrt{-g}F^a_{\mu\nu}F^{\mu\nu a} - \frac{N^2-1}{96\pi^2}\int d^5x\, d^{abc}\varepsilon^{\mu\nu\lambda\rho\sigma}A^a_\mu \partial_\nu A^b_\lambda \partial_\rho A^c_\sigma. \tag{3.2}$$

In 5D, the Chern-Simons (CS) term is cubic in fields. Thus, two point correlation functions can be calculated without this term, but three and higher n-point functions receive anomalous contributions from the CS term. The three-point functions can, in principle, be computed from the closed-time-path AdS/CFT prescription. In this paper, we compute two-point functions in the presence of a background magnetic field — which couples to the two-point functions through the CS term in the 5D action,

In ref. [5] the general idea, originated from the black-hole membrane paradigm, begins with defining conserved currents in terms of field tensors evaluated on a stretched horizon. The prescription for how to define these currents is lifted directly from the membrane paradigm [6]. Then, one takes the long wavelength limit of the field equations in order to derive the constitutive equation for the currents. It was shown in ref. [5] that the 5D Maxwell equations imply Fick's law from boundary currents, with a diffusion constant matching that found by direct AdS/CFT calculation in the case of $\mathcal{N} = 4$ SYM.

Working in the classical regime of the higher dimensional theory, we use the abelian field strength and ignore the f^{abc} terms in the Yang-Mills action. The modified Maxwell equations obtained from our action are,

$$\frac{1}{g_{\rm SG}^2\sqrt{-g}}\,\partial_\nu[\sqrt{-g}F^{a\mu\nu}] + \frac{N^2\, d^{abc}}{128\pi^2\sqrt{-g}}\varepsilon^{\mu\lambda\nu\rho\sigma}F^b_{\lambda\nu}F^c_{\rho\sigma} = 0. \tag{3.3}$$

In order to simplify our task, let us turn on a constant, homogeneous background magnetic field, $\mathbf{B}$, in the 4D theory and discuss charge diffusion on this background. Turning on $\mathbf{B}$ in 4D means imposing a boundary condition that the 5D field strength approach $\mathbf{B}$ as $u \to 0$. One can see that these equations support a constant magnetic field in the three space-like dimensions perpendicular to the branes,

$$-\frac{1}{2}\varepsilon^{ijk}F_{jk} = B^i. \tag{3.4}$$

We will neglect the back-reaction of this field on the background metric. This is justified when $B \ll T^2$. Later on, we will work in $A_5 = 0$ gauge and make use of a partial Fourier decomposition of the A_μ into plane waves parallel to the horizon with dependence on the fifth coordinate left explicit:

$$A_\mu(u, \omega, \mathbf{q}) = \int d^4x A_\mu(u, t, \mathbf{x}) e^{i\omega t - i\mathbf{q}\cdot\mathbf{x}}.$$

Throughout, we will use the notation

$$\tilde{B} = \frac{B}{2(\pi T)^2}\,; \qquad \tilde{\omega} = \frac{\omega}{2\pi T}\,; \qquad \tilde{q} = \frac{q}{2\pi T}.$$

In addition, we will make the assumptions $\tilde{\omega} \sim \tilde{q}^2 \sim \tilde{B}^2$ and $\tilde{q} \ll 1$ which in the end are seen to be consistent with the modified diffusion equation we obtain.

The solution to the field equations describing a constant homogeneous magnetic field, together with the metric (3.1), defines the 5D classical background about which we linearize in small perturbations. The linearized Maxwell equations become:

$$\frac{1}{g_{\rm SG}^2\sqrt{-g}}\,\partial_\nu[\sqrt{-g}g^{5\rho}g^{\nu\sigma}F^a_{\rho\sigma}] - \frac{N^2\,d^{abc}}{16\pi^2\sqrt{-g}}B^b_i F^c_{ti} = 0, \tag{3.5}$$

$$\frac{1}{g_{\rm SG}^2\sqrt{-g}}\,\partial_\nu[\sqrt{-g}g^{t\rho}g^{\nu\sigma}F^a_{\rho\sigma}] + \frac{N^2\,d^{abc}}{16\pi^2\sqrt{-g}}B^b_i F^c_{5i} = 0, \tag{3.6}$$

$$\frac{1}{g_{\rm SG}^2\sqrt{-g}}\,\partial_\nu[\sqrt{-g}g^{i\rho}g^{\nu\sigma}F^a_{\rho\sigma}] - \frac{N^2\,d^{abc}}{16\pi^2\sqrt{-g}}B^b_i F^c_{5t} = 0. \tag{3.7}$$

In a theory with anomaly, the definition of current becomes ambiguous. We shall require that all currents be gauge invariant under the abelian subgroup of SU(4) singled out by the magnetic field. Thus, we employ the regularization where anomalies appear only in R-charge currents for which no external gauge field is turned on. In this regularization the divergence of an anomalous current is [7]:

$$\partial_\mu j^\mu_a = d^{abc}\frac{N^2-1}{128\pi^2}\varepsilon^{\alpha\beta\gamma\delta}F^b_{\alpha\beta}F^c_{\gamma\delta}. \tag{3.8}$$

In ref. [5] the radial Maxwell equation, (3.5) is found to function as a conservation equation for the membrane paradigm currents. In our case, we define currents for which the radial Maxwell equation serves as the Adler-Bell-Jackiw anomaly equation, (3.8). Ultimately, we will show that when we take

$$j^t_a \;=\; -\frac{\sqrt{-g_{5D}}}{g^2_{SG}}F^{5t}_a|_{u_{sh}}, \tag{3.9}$$

$$j^i_a \;=\; -\frac{\sqrt{-g_{5D}}}{g^2_{SG}}(F^{5i}_a - d^{abc}\tilde{B}^i_b F^{5t}_c)|_{u_{sh}}. \tag{3.10}$$

to define the currents for our system, equation (3.8) is properly satisfied. One can already see that when $\mathbf{B} = \mathbf{0}$ these currents are conserved by the Maxwell equation. The second term on the right hand side of (3.10) is a modification to Fick's law, arising from the presence of the Chern-Simons term in the action.

3.2 Analysis

We now embark on our study of linearized perturbations about our solution. Our general strategy will be to demonstrate the same two essential facts in [5]. First, as a direct result of incoming wave boundary conditions at the horizon

$$F_{5i}(u_{sh}) \propto F_{ti}(u_{sh}).$$

Second, in the hydrodynamic regime

$$F_{ti}(u_{sh}) \approx -\partial_i A_t(u_{sh}). \tag{3.11}$$

Our analysis differs from [5] - and not only by our inclusion of new terms in the equations of motion and our accommodation of a background **B** field. We also must employ different boundary conditions on A_t in order to allow for a non-vanishing **E** in the boundary theory. In [5], $A_t \to 0$ at the boundary leads, through (3.11) to $F_{ti} \approx -\partial_i F_{5t}$, which supplies the diffusion term. In our case, a non-zero $A_t(u=0)$ leads, in the same manner, to diffusion and Ohmic terms in the constitutive relation, as well as an $\mathbf{E}\cdot\mathbf{B}$ non-conservation term for an anomalous current.

We make use of three Maxwell equations (depending on whether the free index is spatial, temporal, or radial) plus two Bianchi Identities:

$$\partial_t F^a_{5t} - (1-u^2)\partial_j F^a_{5j} - d^{abc}\tilde{B}^b_j F^c_{tj} = 0, \tag{3.12}$$

$$\partial_5 F^a_{5t} - \frac{1}{(2\pi T)^2 u(1-u^2)}\partial_j F^a_{tj} + d^{abc}\tilde{B}^b_j F^c_{5j} = 0, \tag{3.13}$$

$$\partial_5[(1-u^2)F^a_{5i}] - \frac{1}{(2\pi T)^2 u(1-u^2)}\partial_t F^a_{ti} - \frac{1}{(2\pi T)^2 u}\partial_j F^a_{ij} + d^{abc}\tilde{B}^b_i F^c_{5t} = 0, \tag{3.14}$$

$$\partial_t F_{5j} - \partial_j F_{5t} - \partial_5 F_{tj} = 0, \tag{3.15}$$

$$\partial_i F_{tj} - \partial_j F_{ti} - \partial_t F_{ij} = 0. \tag{3.16}$$

As in [5] a single wave equation for F_{ti} can be obtained in the near horizon limit. Combining (3.15) with (3.14) and (3.12) respectively we find

$$\frac{\partial_t^2 F^a_{ti}}{(2\pi T)^2 u(1-u^2)} - \partial_5[(1-u^2)(\partial_i F^a_{5t} + \partial_5 F^a_{ti})] + \frac{\partial_j\partial_t F_{ij}}{(2\pi T)^2 u} - d^{abc}\tilde{B}^b_i \partial_t F^c_{5t} = 0, \tag{3.17}$$

$$\partial_t^2 F^a_{5t} - (1-u^2)\partial_j(\partial_j F^a_{5t} + \partial_5 F^a_{tj}) + d^{abc}\tilde{B}^b_j \partial_t F^c_{tj} = 0. \tag{3.18}$$

Near horizon, these equations simplify significantly. We proceed under the assumption that all three terms of (3.12) are of the same degree of singularity as we approach the horizon, and check the consistency of this assumption after the fact. This allows us to neglect $\partial_j^2 F_{5t}$ in (3.18) and, passing to momentum space, we find

$$F^a_{5t} \sim -i(1-u^2)\frac{q}{\omega^2}\partial_5 F^a_{tj} + \frac{1}{\omega^2} d^{abc}\tilde{B}^b_j \partial_t F^c_{tj}. \tag{3.19}$$

It then follows that for $1-u \ll \omega^2/q^2$ the F_{5t} terms in (3.17) can be omitted. The Bianchi identity (3.16) indicates that the F_{ij} term can be omitted as well, and we obtain a wave equation for F_{ti}:

$$\partial_t^2 F_{ti} - (2\pi T)^2 u(1-u^2)\partial_5[(1-u^2)\partial_5 F^a_{ti}] = 0. \tag{3.20}$$

Near horizon, this is solved by

$$F_{ti}(u,t) = [\alpha_i(1-u)^{\frac{i\tilde{\omega}}{2}} + \beta_i(1-u)^{-\frac{i\tilde{\omega}}{2}}]e^{i\omega t}. \tag{3.21}$$

If we specify incoming wave boundary conditions at the horizon, only the first term can be allowed to contribute, and taking time and radial derivatives of the last equation gives us

$$\partial_t F^a_{ti} - (4\pi T)(1-u)\partial_5 F_{ti} = 0. \tag{3.22}$$

Bianchi identity (3.15) then indicates, through (3.19) that

$$\partial_t \left[F_{5i} - \frac{F_{ti}}{(4\pi T)(1-u)} \right] = 0. \tag{3.23}$$

Since finite energy solutions must decay with time, we have

$$F_{5i} = \frac{F_{ti}}{(4\pi T)(1-u)}. \tag{3.24}$$

Note that our results for F_{ti} and F_{5i} are consistent with the assertion that all terms of (3.12) are comparably divergent as we approach the horizon. We have now established a relationship between $F_{5i} \subset j_i$ and F_{ti}. To demonstrate a modified Fick's law, we must still relate this to a gradient of F_{5t}. This is particularly straight forward in the $A_5 = 0$ gauge, as $F_{5t} = \partial_5 A_t$. Below, we have separated A_i into two pieces: $A_i = A_i^*(\mathbf{x}) + A_i(u,t,\mathbf{x})$ where A_i^* gives rise to the constant magnetic field, and $A_i(u,t,\mathbf{x})$ is the arbitrarily weak perturbation about the classical background. In the hydrodynamic regime, and for a weak enough magnetic field, it is possible to find solutions for $A_t(u)$ and $A_i(u)$, perturbatively in $\tilde{\omega}$, $\tilde{q}$, and $\tilde{B}$, such that

$$\frac{A_t|_{sh} - A_t|_0}{\partial_5 A_t|_{sh}} \approx constant, \tag{3.25}$$

while

$$F_{ti}|_{sh} \approx -\partial_i A_t|_{sh}. \tag{3.26}$$

Specifically, we take $\tilde{q} \ll 1$ and $\tilde{\omega} \sim \tilde{q}^2 \sim \tilde{B}^2$.

Finding the perturbative solutions to first order in the small quantities is no simple matter for general d^{abc}, but it will not be necessary for our purposes. We will simply show that the leading terms of both solutions satisfy the above conditions, determine the value of the constant, and demonstrate that corrections to these leading terms are small enough that the procedure is valid. As in [5] we take the stretched horizon to be close enough to the actual horizon to satisfy $1 - u_{sh} \ll 1$ without being exponentially close:

$$-\tilde{\omega}\ln(1-u_{sh}) \ll 1. \tag{3.27}$$

Then, all the way down to the stretched horizon we can take

$$(1-u)^{\frac{i\tilde{\omega}}{2}} \approx 1. \tag{3.28}$$

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To isolate dependence on radial coordinate, we write the momentum space Maxwell equations, using a prime to denote radial differentiation:

$$\omega\, A_t^{a\prime} + (1-u^2)q_j A_j^{a\prime} + d^{abc}\tilde{B}_j^b(q_j A_t^c + \omega A_j^c) \,=\, 0, \tag{3.29}$$

$$A_t^{a\prime\prime} - \frac{1}{(2\pi T)^2 u(1-u^2)}q(qA_t^a + \omega A_j^a) + d^{abc}\tilde{B}_j^b A_j^{c\prime} \,=\, 0, \tag{3.30}$$

$$\begin{aligned}[(1-u^2)A_i^{a\prime}]' &+ \frac{\omega}{(2\pi T)^2 u(1-u^2)}(qA_t^a + \omega A_j^a) \\ &+ \frac{q_j}{(2\pi T)^2 u}(q_i A_j^a - q_j A_i^a) + d^{abc}\tilde{B}_i^b A_t^{c\prime} \,=\, 0.\end{aligned} \tag{3.31}$$

Then, solving equations (3.30) and (3.31) for $\tilde{\omega} = \tilde{q} = \tilde{B} = 0$ with the boundary conditions $A_i(0) = 0$, $A_t 0 = const.$, $A_\mu(u=1) = const.$ we find

$$A_t^{a(0)}(u) \,=\, uC_t^{a(0)} + A_t^{a(0)}(0), \tag{3.32}$$

$$(1-u^2)A_j^{a(0)\prime} \,=\, C_j^{a(0)}. \tag{3.33}$$

Thus the constant relating A_t to $\partial_5 A_t$ is one. Now, if we substitute (3.24) into (3.29) (keeping in mind we need only near horizon results to make statements concerning the currents) we can relate $C_j^{a(0)}$ to $C_t^{a(0)}$.

$$\omega A_t^{a\prime} + (1-u^2)q_j A_j^{a\prime} - i(2\pi T)d^{abc}\tilde{B}_j^b[(1-u^2)A_j^{c\prime}] = 0. \tag{3.34}$$

In fact, we have a matrix equation,

$$A_t^{a\prime} = \frac{1}{\omega}[q_j\delta^{ac} - i(2\pi T)d^{abc}\tilde{B}_j^b][(1-u^2)A_j^{c\prime}], \tag{3.35}$$

that could be used to find $A_j^{(0)}$ from $A_t^{(0)}$ — though we need not do so here. It is important to our argument that this matrix equation be non-singular. While we are not aware of any generally applicable reason it should not be, we can always restrict our analysis to some subgroup of the R-charge SU(4) for which this will be true regardless of the relative value of $\tilde{q}$ and $\tilde{B}$. Thus, keeping to our assumption that $\tilde{\omega} \sim \tilde{q}^2 \sim \tilde{B}^2$, we have $C_j^{a(0)} \sim \frac{\omega}{q}C_t^{a(0)}$ and

$$A_j^{a(0)} \sim A_t^{a(0)}\frac{\omega}{q}\ln\left(\frac{1+u}{1-u}\right). \tag{3.36}$$

Now we substitute $A_t = A_t^{(0)} + A_t^{(1)}$, $A_j = A_j^{(0)} + A_j^{(1)}$ into equation (3.30), again exploiting (3.24), and find

$$A_t^{a(1)\prime\prime} = 2\tilde{q}_j A_j^{a(0)\prime} - d^{abc}\tilde{B}_j^b A_j^{c(0)\prime}. \tag{3.37}$$

Hence,

$$A_t^{(1)\prime\prime} \sim \frac{\tilde{\omega}}{(1-u^2)}A_t^{(0)}, \tag{3.38}$$

so,

$$A_t^{(1)} \sim \tilde{\omega}A_t^{(0)}. \tag{3.39}$$

Finally, invoking (3.29) again, we see

$$A_j^{(1)} \sim A_t^{(1)}\frac{\omega}{q}\ln\left(\frac{1+u}{1-u}\right). \tag{3.40}$$

Thus, we can safely conclude that $F_{ti}|_{sh} \approx -\partial_i A_t|_{sh}$.

3.3 Summary

In this way, we find ourselves in possession of solutions satisfying (3.25) and (3.26). We now see that defining currents as in (3.10) allows (3.12) to function as the continuity equation. Application of (3.24) now results in a modified Fick's Law:

$$\mathbf{j}^a = -D\boldsymbol{\nabla}\rho^a + \sigma\,\mathbf{E}^a + d^{abc}\tilde{\mathbf{B}}^b\rho^c. \tag{3.41}$$

Here, $D = \frac{1}{2\pi T}$ is the diffusion constant, while $\chi = \frac{N^2T^2}{8}$ is the susceptibility and $\sigma = \frac{N^2T}{16\pi}$ the conductivity [8]. As a result, (3.12) becomes a modified diffusion equation:

$$\partial_t\rho^a - D\nabla^2\rho^a + \sigma\boldsymbol{\nabla}\cdot\mathbf{E}^a + \left(\frac{N^2}{16\pi^2\chi}\right)d^{abc}\mathbf{B}^b\cdot\boldsymbol{\nabla}\rho^c = d^{abc}\frac{N^2}{16\pi^2}\mathbf{B}^b\cdot\mathbf{E}^c. \tag{3.42}$$

As noted at the beginning of this section, we cannot allow currents coupled to a gauge field to be non-conserved lest the theory be inconsistent. Hence, no one of the fifteen R-charges in the boundary theory will have all of the above terms in its diffusion equation. Charges coupling to a gauge field will have an Ohmic term, but no $\mathbf{E}\cdot\mathbf{B}$ term:

$$\partial_t\rho^a - D\nabla^2\rho^a + \sigma\boldsymbol{\nabla}\cdot\mathbf{E}^a + \left(\frac{N^2}{16\pi^2\chi}\right)d^{abc}\mathbf{B}^b\cdot\boldsymbol{\nabla}\rho^c = 0. \tag{3.43}$$

Anomalous charges will have no Ohmic term:

$$\partial_t\rho^a - D\nabla^2\rho^a + \left(\frac{N^2}{16\pi^2\chi}\right)d^{abc}\mathbf{B}^b\cdot\boldsymbol{\nabla}\rho^c = d^{abc}\frac{N^2}{16\pi^2}\mathbf{B}^b\cdot\mathbf{E}^c. \tag{3.44}$$

4. Constitutive equations at weak coupling

Above, we demonstrated an anomalous modification of constitutive relations in a strongly coupled conformal field theory with a gravitational dual. We now turn to theories at weak coupling and argue that the same modification should also appear there, and explain its physical origin.

To keep the discussion simple, let's consider the theory of a massless fermion. To have a hydrodynamic behavior at finite temperature the fermion should interact with itself, but we shall assume the interaction to be arbitrarily weak. In accordance to the discussion in the previous section, we turn on a background gauge field coupled to the fermion current,

$$A^\mu = (0, 0, Bx, 0).$$

Let us emphasize again that A^μ is only a background gauge field; we do not include a dynamical U(1) gauge field into the theory. The fermionic lagrangian for our theory is

$$\mathcal{L}_\psi = \bar{\psi}\gamma_\mu D^\mu\psi, \tag{4.1}$$

with ψ a four-component Dirac spinor and $D^\mu = \partial^\mu - ieA^\mu$. The fermion hamiltonian can be decomposed into the left- and right-handed parts (Below, $\boldsymbol{\sigma}$ is the vector of Pauli matrices.)

$$H_\psi = i\int d^3x\,(\bar{\psi}_{\mathrm{L}}\boldsymbol{\sigma}\cdot\mathbf{D}\psi_{\mathrm{L}} - \bar{\psi}_{\mathrm{R}}\boldsymbol{\sigma}\cdot\mathbf{D}\psi_{\mathrm{R}}). \tag{4.2}$$

Energy eigenstates of definite chirality can now be found which satisfy $-i\boldsymbol{\sigma}\cdot\mathbf{D}\psi_{\rm R} = E_{\rm R}\psi_{\rm R}$ and $i\boldsymbol{\sigma}\cdot\mathbf{D}\psi_{\rm L} = E_{\rm L}\psi_{\rm L}$. The solutions can be separated via

$$\psi_{\rm R,L}(x,y,z) = \begin{pmatrix} \varphi_{\rm R,L}(x) \\ \phi_{\rm R,L}(x) \end{pmatrix} e^{i(k_y y + k_z z)}. \tag{4.3}$$

We have two pairs of coupled first ordered differential equations for ϕ and φ which can be written as a second order differential equation for any one of them. For example,

$$\left[\nabla_{\bar{x}}^2 - e^2B^2\bar{x}^2 + E^2 - k_z^2 + eB\right]\phi_R = 0. \tag{4.4}$$

where $\bar{x} = x - k_y/eB$, Thus, we may write solutions for $\phi_{\rm R}$ as,

$$\phi_2(\bar{x}, n) = \left(\frac{eB}{(n!)^2 2^{2n}\pi}\right)^{1/4} \mathcal{H}_n\left(\sqrt{eB}\bar{x}\right) e^{-\frac{eB}{2}\bar{x}^2}. \tag{4.5}$$

Here, $\mathcal{H}_n(x)$ is the Hermite polynomial defined by

$$\mathcal{H}_n(x) = (-1)^n e^{x^2}\frac{d^n}{dx^n}e^{x^2}. \tag{4.6}$$

The energy spectrum is $E_{\rm R}(n) = \pm\sqrt{k_z^2 + 2neB}$, where n is the Landau level's label. There is, however, a subtlety at $n = 0$. The equation from which $\varphi_{\rm R}$ must now be determined is,

$$[k_z - E_{\rm R}(n)]\varphi_{\rm R} = i(\nabla_{\bar{x}} + eB\bar{x})\phi_{\rm R}. \tag{4.7}$$

When $n = 0$ this equation can only be solved if $E_{\rm R}(0) = -k_z$. Thus, the energy eigenstates with $n = 0$ are chiral: the right-handed excitations form a single branch $E = -k_z$, while the left-handed energy levels are $E = k_z$.

In the vacuum, all negative energy states are filled. If one turns on a chemical potential $\mu > 0$ for the vector charge. then in addition to the Dirac sea all energy levels with $0 < E < \mu$ are populated. For the $n = 0$ energy levels, this means left-handed fermions with $0 < k_z < \mu$ and right-handed fermions with $-\mu < k_z < 0$ are populated. But as all left-handed fermions have positive k_z and right-handed ones have negative k_z, the net result is that there is a nonzero axial current that comes from the $n = 0$ states. One can also check that the $n \neq 0$ states do not contribute to the axial current, since these energy levels are not chiral: the contribution from left- and right-handed sectors cancel each other. The effect survives at finite temperature.

The effect can be quantified most easily by considering a finite volume. If we place the theory in a square box of side length L, with periodic boundary conditions, then the momentum takes discrete values $k_i = \frac{2}{\pi}n_i/L$. Since the energy eigenvalues do not depend on k_y, there is a degeneracy of various k_y states at each value of $E(n)$. Specifically, we require $\bar{x} = 0$ to be inside the box forces $0 \leq k_y < eBL$, which means that each value of n corresponds to $eBL^2/(2\pi)$ possible values of k_y.

In thermal equilibrium the total current is the sum over all energy levels,

$$j^z_{\rm R,L}(\mathbf{x}) = \frac{1}{L^3}\sum_{n=0}^{\infty}\sum_{k_z}\sum_{k_y} E_k f_{\rm R,L}(n, k_z, k_y, \mathbf{x}).$$

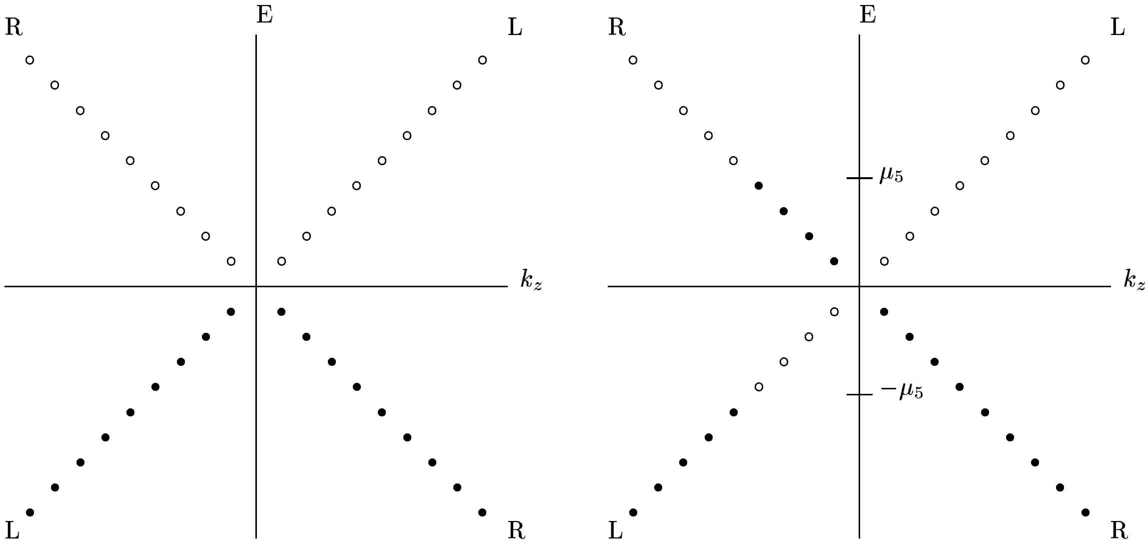

Figure 1: A non-zero electric charge density gives rise to opposing fluxes of left and right handed particles, culminating in $\mathbf{j}_{\mathrm{R-L}} \propto -\mu \mathbf{B}$.

As mentioned above, states with $n \neq 0$ do not contribute to the axial current, since these positive energy states exist in equal number with both signs of k_z, regardless of chirality. The function f is the Fermi-Dirac distribution function,

$$f_{R,L}(k_z, k_y; T, \mu) = \frac{1}{e^{\beta\,[|k_z| - \mu]} + 1}. \tag{4.8}$$

After taking into account the degeneracy factor related to k_y, summing over k_z, and regularizing by subtracting off the contribution of the Dirac sea, one finds

$$\mathbf{j}_5 = \mathbf{j}_{\mathrm{R}} - \mathbf{j}_{\mathrm{L}} = -\frac{e}{2\pi^2}\mu \mathbf{B}, \tag{4.9}$$

which coincides with the result obtained for the gravity dual theory in the previous section.

The same analysis we just used can also be applied in determining the electric current at equilibrium, in the presence of a non-zero chemical for the axial charge μ_5 and homogeneous background $\mathbf{B}$. Just as in the case of $\mathcal{N} = 4$ SYM, we find that the constitutive relation of a non-anomalous current is altered in the same manner as that of the anomalous current. Specifically,

$$\mathbf{j}_{\mathrm{EM}} = -\frac{e}{2\pi^2}\mu_5 \mathbf{B}. \tag{4.10}$$

(See figure 2.) This is the exact same result obtained in [3].

5. Coupling independence

The presence of a new type of constitutive terms in the hydrodynamics of quantum field theories, on the background of a homogeneous magnetic field, is to be expected on the basis of simple symmetry arguments. These terms look like,

$$\mathbf{j}_a = \sum_c C_{abc} j_b^0 \mathbf{B}_c\,.$$

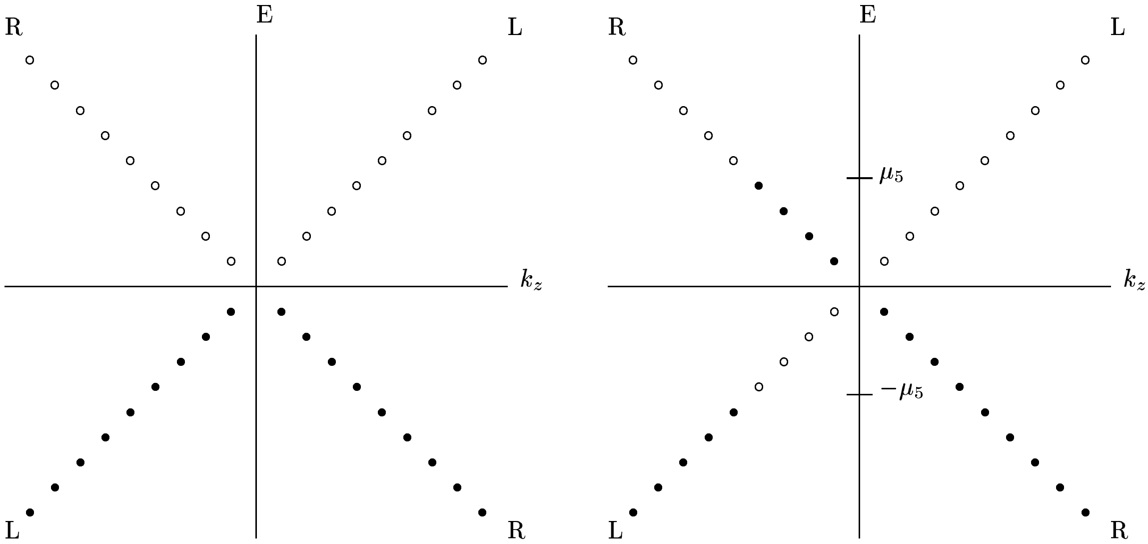

Figure 2: A chiral density composed of equal numbers of right-handed particles and left-handed antiparticles gives rise to a flux of positive charge in the direction of $\mathbf{B}$.

In the case of a strongly t'Hooft coupled, large-N gauge theory dual to an AdS_5 supergravity, we see these terms arise naturally from the R-charge anomaly in a membrane paradigm approach. In this strongly coupled case, the coefficients of the new terms are simply determined from the anomaly coefficient appearing in the divergence equation, and the susceptibility for the charge density in the new term. Specifically,

$$\partial_\mu j^\mu_a = \eta\, d_{abc}\mathbf{E}^b \cdot \mathbf{B}^c$$

indicates that,

$$C_{ac} = -\frac{\eta}{\chi_c} d_{abc}\,.$$

(Since $\mathbf{B}$ is only present for one charge, C need not depend on the index c. Note also, that while the order of indices in unimportant in the tensor structure of the anomaly, it is none the less used in C_{abc} to indicate that we are considering the correction to the current, a, that is proportional to the charge density, b. The factor η is dimensionless, and geometric in origin.) In the case of arbitrarily weakly coupled QED at finite temperature, we see the exact same contribution to currents arise from a local thermal equilibrium treatment, consistent with [3]. Here, the currents arise as a direct result of the effect of the chiral anomaly on the spectrum of the non-interacting theory. In this section we will formulate an argument as to why the coefficients of the new terms should generally be determined as they are in the two, disparate examples we have studied above. We'll do this, by combining linear response theory with hydrodynamics, and demanding that the hydrodynamics be consistent with the zero frequency, zero momentum limit of a thermalized local quantum field theory.

We use linear response theory to express currents and densities in terms of retarded two point correlators, and perturbing source fields [9]. Then, we invoke symmetry arguments to build hydrodynamic equations for the currents. The hydrodynamic equations will then function as equations for the retarded correlators. Solving these equations and setting the energy to zero, we find that enforcing the correct behavior of the retarded two point functions in the zero momentum limit constrains the coefficients of the constitutive terms to be exactly as they are in our two specific cases above. Doing that we assume the theory to contain no massless excitations that could cause the retarded two point functions to be divergent at exactly zero energy and small momentum. Specifically, this will mean that no Goldstone modes or massless, dynamical gauge fields can be present. The fact that the zero momentum limit enforces this constraint indicates that these "kinetic coefficients" for the new terms are actually determined by equilibrium physics, as observed in [3].

We will work with a toy model containing two global charges, one vector and one axial-vector, participating in an anomaly with the same background (vector) magnetic field. Because there are only two global currents participating in the anomaly in this system, we can simplify our notation via $C_{\rm A} = C_{\rm AVV}$ and $C_{\rm V} = C_{\rm VAV}$. The same analysis can be easily extended to a system with a more general set of currents. More will be said about this at the end of the section.

LRT gives the response of a current to the presence of a source field, up to linear order in the source, as

$$\delta j^\mu_a(\omega, \mathbf{p}) = [\Pi^{\mu\nu}_R(\omega, \mathbf{p})]_{ab} A^b_\nu(\omega, \mathbf{p}).$$

Here,

$$[\Pi^{\mu\nu}_R(x-y)]_{ab} = \theta(x^0 - y^0)\langle[j^\mu_a(x), j^\nu_b(y)]\rangle,$$

is the retarded current-current correlator in the absence of the perturbation, and

$$\delta j^\mu_a = \langle j^\mu_a\rangle|_A - \langle j^\mu_a\rangle|_0,$$

is the difference between the current's expectation value in the presence, and absence of the source. Here, we will employ a source for the temporal component of our vector current only. For consistency with the rest of the paper, we will use the notation, $\rho_a = \delta j^0_a$ in this section, thus:

$$\rho_{\rm V} = [\Pi^{00}_{\rm R}]_{\rm VV} A^{\rm V}_0 \quad ; \qquad \rho_{\rm A} = [\Pi^{00}_{\rm R}]_{\rm AV} A^{\rm V}_0 .$$

Substituting the LRT charge fluctuations into our modified diffusion equations gives,

$$(\partial_t - D_{\rm V}\nabla^2)\rho_{\rm V} = -\sigma_{\rm V}\boldsymbol{\nabla}\cdot\mathbf{E}_{\rm V} - C_{\rm V}\mathbf{B}_{\rm V}\cdot\boldsymbol{\nabla}\rho_{\rm A}, \tag{5.1}$$

$$(\partial_t - D_{\rm A}\nabla^2)\rho_{\rm A} = -C_{\rm A}\mathbf{B}_{\rm V}\cdot\boldsymbol{\nabla}\rho_{\rm V} + \eta d_{\rm AVV}\mathbf{B}_{\rm V}\cdot\mathbf{E}_{\rm V}. \tag{5.2}$$

Since $\mathbf{E}_{\rm V} = -\boldsymbol{\nabla}A^0$, passing to momentum space allows us to drop a factor of $\tilde{A}^0(\omega, \mathbf{p})$ from each term in both equations, leaving us with equations for the current-current correlators.

$$(i\omega - D_{\rm V}p^2)[\tilde{\Pi}^{00}_{\rm R}]_{\rm VV} = \sigma_{\rm V}p^2 + iC_{\rm V}\mathbf{p}\cdot\mathbf{B}_{\rm V}[\tilde{\Pi}^{00}_{\rm R}]_{\rm AV}, \tag{5.3}$$

$$(i\omega - D_{\rm A}p^2)[\tilde{\Pi}^{00}_{\rm R}]_{\rm AV} = iC_A\mathbf{p}\cdot\mathbf{B}_{\rm V}[\tilde{\Pi}^{00}_{\rm R}]_{\rm VV} - i\eta d_{\rm AVV}\mathbf{p}\cdot\mathbf{B}_{\rm V}. \tag{5.4}$$

Solving for both correlators while keeping (for consistency) only terms up to linear order in the magnetic field, we obtain

$$[\tilde{\Pi}^{00}_{\rm R}]_{\rm AV} = \frac{iC_{\rm A}\mathbf{p}\cdot\mathbf{B}_{\rm V}[\sigma p^2 - \frac{\eta}{C_{\rm A}}d_{\rm AVV}(i\omega - D_Ap^2)]}{(i\omega - D_{\rm V}p^2)(i\omega - D_{\rm A}p^2)}, \tag{5.5}$$

and,

$$[\tilde{\Pi}^{00}_{\rm R}]_{\rm VV} = \frac{\sigma p^2}{(i\omega - D_{\rm V}p^2)}. \tag{5.6}$$

We are now equipped to ask what zero momentum behavior the hydrodynamic equations instill in the retarded correlators, and to demand that this be consistent with the known behavior from finite temperature field theory. First, we will take $\omega \to 0$ and note that in this limit, the retarded correlator becomes an analytic continuation of the euclidean time (or Matsubara) two point correlator [9]. Now, taking $p \to 0$ we note that $\lim_{p\to 0}[\tilde{\Pi}^{00}_{\rm R}]_{\rm AV}$ is well defined. To examine the momentum dependence of our hydrodynamic $[\tilde{\Pi}^{00}_{\rm R}]_{\rm AV}$ it will be convenient to separate the momentum into components perpendicular and parallel to the magnetic field. Writing $B = |\mathbf{B}_{\rm V}|$ we find,

$$[\tilde{\Pi}^{00}_{\rm R}(0,\mathbf{p})]_{\rm AV} = \left(\frac{p_\parallel}{(p_\parallel^2 + p_\perp^2)}\right)\frac{iC_{\rm A}B(\sigma + \frac{\eta}{C_{\rm A}}D_{\rm V})}{D_{\rm V}D_{\rm A}}. \tag{5.7}$$

Observe, that if we take $p \to 0$ along a contour that keeps the ratio $p_\perp^2/p_\parallel$ constant, the value of $\lim_{p\to 0}[\tilde{\Pi}^{00}_{R}]_{\rm AV}$ will be entirely dependent upon what the value of $p_\perp^2/p_\parallel$ is. Thus, the limit is not well defined, unless the numerator itself is zero. The only way this can happen in the context of our hydrodynamic equations, is for the following equality to hold:

$$C_{\rm A} = -\eta\frac{D_{\rm V}}{\sigma_{\rm V}}d_{\rm AVV}.$$

Taking the same limits for $\tilde{\Pi}^{00}_{\rm R}]_{\rm VV}$ returns the susceptibility, as it should, and gives no information about $C_{\rm A}$ or $C_{\rm V}$. The constant $C_{\rm V}$ can be found by repeating the same steps using only a non-zero A^0_A, and examining $\tilde{\Pi}^{00}_{\rm R}]_{\rm VA}$.

As mentioned earlier, this analysis follows through for more general sets of vector and axial-vector currents as well. The currents are fluxes of global charges, of the form

$$j^\mu_a = \bar{\psi}_i\gamma^\mu V^{ij}_a\psi_j \quad , \quad j^\mu_x = \bar{\psi}_i\gamma^\mu\gamma_5 A^{ij}_x\psi_j,$$

where V_a and A_x are the generators, in flavor space, of the symmetries giving rise to each charge. (We will use early Latin indices for vector current generators, and late Latin indices for axial current generators, for clarity in what follows.) In general, we can expect $[V_a, V_b] \neq 0$, $[A_x, A_y] \neq 0$, and $[V_a, A_x] \neq 0$. The applied magnetic field couples only to one of the vector charges, which we specify here by the index b. In this case, symmetry will demand that new constitutive terms have the forms,

$$\mathbf{j}_x = \sum_b C_{xbc}\rho_b\mathbf{B}_c \quad , \quad \mathbf{j}_a = \sum_c C_{azc}\rho_z\mathbf{B}_c.$$

The equations of motion to be solved will now be matrix equations in the space of the global charges denoted by $\{x,a\}$. Again, one finds that in order for the infrared limit of the two point functions to behave properly,

$$C_{xbc} = -\eta d_{xbc}\chi_b^{-1},$$

must hold for each possible value of x, and c, while

$$C_{azc} = -\eta d_{azc}\chi_z^{-1},$$

holds for each a and z, in the presence of $\mathbf{B}_c$. This line of argument applies regardless of the coupling strength of the theory under consideration.

6. Quark gluon plasma in magnetic field

It would be nice to get a look at the kind of effect our modified hydrodynamics could have in a real physical system. One case in which transport behavior in a magnetized system with anomaly may be of interest is hot QCD in a strong magnetic field [12, 13].

We will take a look at chiral QCD with three massless flavors, in the presence of a homogeneous background $\mathrm{U}(1)_{\mathrm{EM}}$ magnetic field. This theory possesses an axial anomaly coupling the electromagnetic (Q), baryon (b), and chiral EM (5) currents. To be explicit, we write these currents in terms of the diagonal generators of $\mathrm{SU}(3)_f$:

$$j_Q^\mu = e\bar{\psi}_i\gamma^\mu \mathbf{Q}_{ij}\psi_j \tag{6.1}$$

$$j_5^\mu = \bar{\psi}_i\gamma^\mu\gamma_5 \mathbf{Q}_{ij}\psi_j \tag{6.2}$$

$$j_b^\mu = \bar{\psi}_i\gamma^\mu \mathbf{b}_{ij}\psi_j \tag{6.3}$$

Where,

$$\mathbf{Q} = \frac{1}{3}\begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad ; \qquad \mathbf{b} = \frac{1}{3}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad ; \qquad \psi = \begin{pmatrix} u \\ d \\ s \end{pmatrix}.$$

There are two more currents diagonal in flavor which may be defined as

$$j_\lambda^\mu = \bar{\psi}_i\gamma^\mu \boldsymbol{\lambda}_{ij}\psi_j \quad , \quad j_{\lambda 5}^\mu = \bar{\psi}_i\gamma^\mu\gamma^5 \boldsymbol{\lambda}_{ij}\psi_j,$$

with

$$\boldsymbol{\lambda} = \frac{1}{2}\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

These currents have hydrodynamic behavior, and are anomalous, but their hydrodynamic equations do not couple to the other currents, since $Tr[\boldsymbol{\lambda}\boldsymbol{\lambda}\mathbf{Q}] = 0$. Thus, we ignore them. The anomaly structure generates non-vanishing three point functions of the types, $\langle j_Q^\nu j_Q^\lambda j_5^\rho\rangle$, and $\langle j_Q^\nu j_b^\lambda j_5^\rho\rangle$.

In order that we be able to apply the methods of the last section, it is important that we take the system to be above the temperature at which the chiral phase transition takes place, so that no pion modes will be present to couple to the chiral current. Furthermore, we must treat the $\mathrm{U}(1)_{EM}$ coupling as arbitrarily small, so the effects of dynamical gauge fields will be negligible. Corrections due dynamic electromagnetic fields would be of order $\frac{\alpha_{\mathrm{EM}}}{\alpha_s}$. Under these conditions, the electromagnetic current is diffusive, rather than ohmic.

Leptons are considered as absent. Thus, all three currents are carried only by quarks, and will have identical diffusion constants.

If we apply the methods of the last section to this model, and again work only to first order in the applied magnetic field, we find the following set of coupled hydrodynamic equations. (To keep notation tidy, we will use $\mathbf{B}' \equiv \frac{e\mathbf{B}}{2\pi^2\chi_{\mathrm{Q}}}$ where $\mathbf{B}$ is the standard magnetic field.)

$$(\partial_t - D\nabla^2) j_Q^0 = -\frac{2}{9}\mathbf{B}' \cdot \nabla j_5^0, \tag{6.4}$$

$$(\partial_t - D\nabla^2) j_5^0 = -\frac{2}{9}\mathbf{B}' \cdot \nabla j_b^0 - \frac{2}{9}\mathbf{B}' \cdot \nabla j_{\mathrm{Q}}^0, \tag{6.5}$$

$$(\partial_t - D\nabla^2) j_b^0 = -\frac{2}{9}\frac{\chi_{\mathrm{Q}}}{\chi_b}\mathbf{B}' \cdot \nabla j_5^0. \tag{6.6}$$

We take the electric field to be zero here. Thus all three charges are conserved. Passing to momentum space, we find a relatively simple eigenvalue equation for the dispersion relations,

$$\begin{pmatrix} (i\omega - D_b q^2) & 0 & -i\frac{2}{9}\mathbf{p} \cdot \mathbf{B}' \\ 0 & (i\omega - D_{\mathrm{Q}} q^2) & -i\frac{2}{9}\mathbf{p} \cdot \mathbf{B}' \\ -i\frac{2}{9}\frac{\chi_{\mathrm{Q}}}{\chi_b}\mathbf{p} \cdot \mathbf{B}' & -i\frac{2}{9}\mathbf{p} \cdot \mathbf{B}' & (i\omega - D_{\mathrm{Q}} q^2) \end{pmatrix} \begin{pmatrix} \rho_b \\ \rho_{\mathrm{Q}} \\ \rho_5 \end{pmatrix} = 0.$$

Clearly, for modes in which the momentum is perpendicular to the magnetic field, the dispersion will be purely diffusive. This will also be the case, when $\frac{B}{T^2} \ll \frac{q}{T}$, regardless of orientation. Modes with momentum parallel to the magnetic field will exhibit more interesting behavior. Specifically we find the new (normalized) eigenmodes to be,

$$\rho_1 \equiv \sqrt{\frac{1}{1 + \chi_{\mathrm{Q}}/\chi_b}} \left(-\frac{1}{\chi_{\mathrm{Q}}/\chi_b}\rho_b + \rho_{\mathrm{Q}} \right), \tag{6.7}$$

$$\rho_2 \equiv \frac{1}{\sqrt{3 + (\chi_{\mathrm{Q}}/\chi_b)}} \left(\rho_b + \rho_{\mathrm{Q}} + \sqrt{1 + (\chi_{\mathrm{Q}}/\chi_b)}\rho_5 \right), \tag{6.8}$$

$$\rho_3 \equiv \frac{1}{\sqrt{3 + (\chi_{\mathrm{Q}}/\chi_b)}} \left(-\rho_b - \rho_{\mathrm{Q}} + \sqrt{1 + (\chi_{\mathrm{Q}}/\chi_b)}\rho_5 \right), \tag{6.9}$$

with dispersion relations,

$$\omega_1 = -iDp^2, \tag{6.10}$$

$$\omega_2 = -iDp^2 - \frac{2}{9}\sqrt{1 + (\chi_{\mathrm{Q}}/\chi_b)}\mathbf{p} \cdot \mathbf{B}', \tag{6.11}$$

$$\omega_3 = -iDp^2 + \frac{2}{9}\sqrt{1 + (\chi_{\mathrm{Q}}/\chi_b)}\mathbf{p} \cdot \mathbf{B}'. \tag{6.12}$$

Frequency eigenmodes are a mixture of currents defined in equation (6.1)–(6.3), and the eigenfrequencies now have a real component with the same form as the dispersion of Alfven waves. (The asymptotic value of $\chi_{\mathrm{Q}}/\chi_b = 2/9$ may be substituted in, if exact numbers are desired.) An interesting effect of this result, is that an over density of $\rho_b + \rho_{\mathrm{Q}}$ will give rise to a dissipationless flow of this charge along the direction of the magnetic field.

7. Conclusions

We have revisited the familiar "level crossing" picture of the impact of a chiral anomaly on the non-interacting limit of a weakly coupled gauge theory. There, we saw that the currents participating in the anomaly are altered in a certain way, in the presence of a homogeneous magnetic field: $\mathbf{j}^a = -\frac{e}{2\pi^2}\mathbf{B}^b\mu^c$. We have examined the hydrodynamics of a strongly coupled (and strongly t'Hooft coupled) plasma using the Gauge-Gravity duality as a tool. There, we saw the R-symmetry currents of large-N, $\mathcal{N}=4$ SYM receive a new term in their constitutive relations, due to their participation in the anomaly: $\mathbf{j}^a = -\frac{N^2}{16\pi^2\chi_c}d^{abc}\mathbf{B}^b\rho^c$.

There is a general symmetry argument for the inclusion of such terms in the constitutive relations of any anomalous local quantum field theory. In the absence of massless modes coupling to the relevant currents, we have a means of determining the coefficients of these terms by demanding consistency between hydrodynamics and linear response theory in the zero momentum limit. Thus, we conclude that, in the presence of static magnetic fields, the hydrodynamic constitutive relations of an anomalous QFT receive a contribution, of the form $d^{abc}\mathbf{B}^b\rho^c$ with a coefficient of $\frac{1}{\chi_c}$ times a geometric factor that can be read off from an anomaly equation. This holds regardless of coupling strength, and so long as their are no massless dynamical modes coupling to the currents involved.

If this LRT treatment can be extended to systems with massless modes, it would be possible to consider anomalous contributions to magnetohydrodynamics. This is an interesting topic for future work, that may significantly broaden the applicability of the considerations raised in this paper.

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